

# STAT 5010 Tutorial 1\*

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## Definitions

1. For  $X, \{X_n\}$  random variables,  $F, \{F_n\}$  the corresponding distribution functions,

- $X_n \xrightarrow{a.s.} X: \mathbf{P}(\lim_{n \rightarrow \infty} X_n = X) = 1$
- $X_n \xrightarrow{L^r} X: \lim_{n \rightarrow \infty} \mathbf{E}|X_n - X|^r = 0$
- $X_n \xrightarrow{p} X: \forall \delta > 0, \lim_{n \rightarrow \infty} \mathbf{P}(|X_n - X| > \delta) = 0$
- $F_n \xrightarrow{w} F: \forall x$  continuity point of  $F, \lim_{n \rightarrow \infty} F_n(x) = F(x)$
- $X_n \xrightarrow{d} X: F_n \xrightarrow{w} F$

2. Let  $X_1, X_2, \dots$  be random vectors and  $Y_1, Y_2, \dots$  be random variables defined on a common probability space.

- $X_n = O_p(Y_n)$  if and only if,  $\forall \epsilon > 0, \exists C_\epsilon > 0$  such that  $\sup_n \mathbf{P}(\|X_n\| \geq C_\epsilon |Y_n|) < \epsilon$ .
- $X_n = o_p(Y_n)$  if and only if  $X_n/Y_n \xrightarrow{p} 0$

## Propositions and Theorems

1. Chebyshev's Inequality:  $\forall t > 0, \mathbf{P}(|X - EX| \geq t) \leq \frac{\text{Var}(X)}{t^2}$

*Proof.*  $t^2 \mathbf{P}(|X - EX| \geq t) = t^2 \mathbf{P}(|X - EX|^2 \geq t^2) = t^2 \mathbf{E}[I_{|X-EX|^2 \geq t^2}] \leq \mathbf{E}[|X - EX|^2] = \text{Var}(X)$

2.  $X_n \xrightarrow{a.s.} X \Leftrightarrow \forall \epsilon > 0, \lim_{k \rightarrow \infty} \mathbf{P}(\bigcup_{n=k}^{\infty} \{|X_n - X| > \epsilon\}) = 0$

*Proof.*  $\mathbf{P}(\lim_{n \rightarrow \infty} X_n = X) = \mathbf{P}(\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}) = \lim_{k \rightarrow \infty} \mathbf{P}(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}) = 1$ .

Note that  $A_k = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| < \frac{1}{k}\}$  are nonincreasing in  $k$ . The equations above hold if and only if  $\mathbf{P}(A_k) = 1$  for all  $k$ , which is equivalent to the statement in the right hand side.

3.  $X_n \xrightarrow{L^r} X \Rightarrow X_n \xrightarrow{p} X$

*Proof.* Similar to the proof of Chebyshev's Inequality.

4.  $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X$

*Proof.* The result follows directly from the proposition 2 above.

5.  $X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$

*Proof.* Let  $x$  be a continuity point of  $F, \epsilon > 0$  given,

$$\begin{aligned} F(x - \epsilon) &= \mathbf{P}(X \in (-\infty, x - \epsilon]) \\ &\leq \mathbf{P}(X \in (-\infty, x - \epsilon], X_n \notin (-\infty, x]) + \mathbf{P}(X_n \in (-\infty, x]) \\ &\leq F_n(x) + \mathbf{P}(|X_n - X| > \epsilon) \end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain that  $F(x - \epsilon) \leq \liminf_n F_n(x)$ .

Similarly, we have  $F(x + \epsilon) \geq \limsup_n F_n(x)$ . Since  $\epsilon$  is arbitrary and  $F$  is continuous at  $x$ ,

$$F(x) = \lim_{n \rightarrow \infty} F_n(x)$$

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6. The corresponding converses of propositions 3-5 are all incorrect.

*Proof.* Construction of counter examples.

7.  $X_n \xrightarrow{p} c \Leftrightarrow X_n \xrightarrow{d} c$ , here  $c$  is a constant.

8. Slutsky's Theorem: For random variables  $X_n, Y_n$  on a probability space,  $X_n \xrightarrow{d} c, Y_n \xrightarrow{d} Y$ , where  $c$  is a fixed real number. Then  $X_n + Y_n \xrightarrow{d} c + Y$ .

## Question 1

1. If random variables  $X_n = O_p(1), Y_n = o_p(1)$ , prove that  $X_n Y_n = o_p(1)$ .

**Solution:** See the handout later.

## Question 2

1.  $X_1, X_2, \dots, X_n, \dots$  are a series of i.i.d. random variables following  $Uniform[0, 1]$  and defined on the same probability space. Give the asymptotic distribution of  $n(1 - X_{(n)})$ .

**Solution:** See the handout later

2. Is it possible to have a random variable  $Y$  s.t.  $n(1 - X_{(n)}) \xrightarrow{a.s.} Y$ ?

**Solution:** See the handout later